Some Notes on Inequalities

Here are four theorems that might really be useful when you're working on an Olympiad problem that involves inequalities. There are a bunch of obscure ones (Chebycheff, Holder, Minkowski, Young, etc.), which are virtually **never** applicable, so we'll just stick to four. You'll see that when you correctly use one of these theorems, a brutally tough question can reveal itself to have an elegant six-line solution

The Fab Four of Inequality Theorems: ← well, really the Fab Three, since ① is just a special case of ③, but oh well. It's all good.

1) **AM-GM** (arithmetic mean - geometric mean inequality)

If $a_1, a_2, \dots, a_n$ are non-negative real numbers, then $\underbrace{\frac{a_1+a_2+\cdots+a_n}{n}}_{\text{"arithmetic" mean}} \geq \underbrace{\sqrt[n]{a_1a_2\cdots a_n}}_{\text{"geometric" mean}}$,

with equality occurring $\underbrace{\text{if and only if}}_{\text{"iff"}}$ $a_1=a_2=\cdots=a_n$.

2) **Cauchy** (or Cauchy-Schwarz-Bunjakovsky if you really want to get technical :)).

If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are non-negative real numbers, then

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2$$

with equality occurring iff $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$.

3) **Power-Mean**

Let $f(x)=\left(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}\right)^{\frac{1}{x}}$, where $a_1, a_2, \dots, a_n$ are non-negative real numbers, and $n \geq 1$. Suppose x and y are integers with $x \geq y$. Then, $f(x) \geq f(y)$, with equality occurring iff $a_1=a_2=\cdots=a_n$.

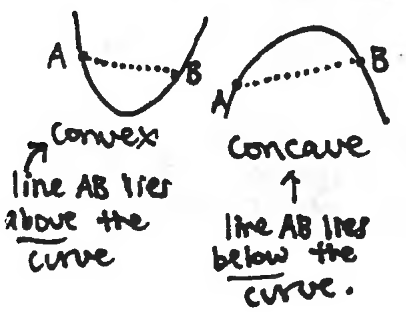
For example, the QM-AM-GM-HM inequality is derived from the Power Mean since $f(2) \geq f(1) \geq f(0) \geq f(-1)$:

$$\underbrace{\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}}_{\text{"quadratic mean"}} \geq \frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n} \geq \underbrace{\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}}_{\text{"harmonic mean"}}$$

Actually, $f(0)$ doesn't exist, but as $x \to 0$, $f(x)$ approaches $\sqrt[n]{a_1a_2\cdots a_n}$.

④ Jensen's Inequality

This is the really cool one. Suppose that $f(x)$ is a real continuous function that is convex (i.e., concave up) on an interval. You test for convexity by showing



that $f''(x) \geq 0$ for all x in that interval → the double or second derivative if you've ever taken calculus. An easier way to see it is if you pick any two points on the curve, and join them. If the line lies above (or on) the curve, it is convex. If the line lies below, then it is concave. (see diagr

i.e. concave down

Let $a_1, a_2, \ldots, a_n$ be n real numbers in an interval S where $f(x)$ is convex for all x in S. Then, $\dfrac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n} \geq f\left(\dfrac{a_1+a_2+\cdots+a_n}{n}\right)$.

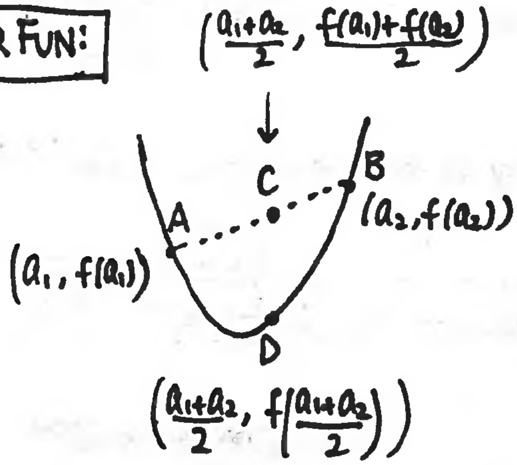
with equality occurring if and only if $a_1 = a_2 = \cdots = a_n$.

If $f(x)$ is concave for all x in S, then $\dfrac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n} \leq f\left(\dfrac{a_1+a_2+\cdots+a_n}{n}\right)$,

same thing, except now the sign is reversed. Same conditions for equality as well (i.e., $a_1 = a_2 = \cdots = a_n$).

These might be quite confusing, especially the last one, so let's try some examples of how we can use these inequalities to solve really challenging problems. The last four are from recent Olympiad contests.

JUST FOR FUN:



Case for $\underline{n=2}$ of Jensen's below:

Let $A(a_1, f(a_1))$, and $B(a_2, f(a_2))$ be any two points on the function $f(x)$, where the function is convex.

Let C be the midpoint of the line AB, and D be the point indicated.

The x-coordinates of C and D are the same but C is above D, i.e.

$$\boxed{\frac{f(a_1)+f(a_2)}{2} \geq f\left(\frac{a_1+a_2}{2}\right)}$$

equality occurs iff $a_1 = a_2$, i.e. A and B are the same point

] What is the minimum possible positive value of $x+\frac{9}{x}$?

learly, we don't want x to be negative. If $x>0$, then both x and $\frac{9}{x}$ are positive, so by the AM-GM inequality, $$\frac{x+\frac{9}{x}}{2} \geq \sqrt{x\cdot\frac{9}{x}} = \sqrt{9} = 3.$$

Thus, $x+\frac{9}{x} \geq 6$, with equality occurring iff $x=\frac{9}{x}$, i.e. $x^2=9$ or $x=3$ (note: $x \neq -3$, since x is positive). We see that indeed, when $x=3$, we have $x+\frac{9}{x} = 3+3 = 6$, thus the minimum possible value of $x+\frac{9}{x}$ is $\underline{\underline{6}}$.

] Prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ if $a,b,c>0$.

olution 1: Expanding, we have $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) = 3+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$.

By AM-GM, since $a,b,c>0$, we have $$\frac{\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}}{6} \geq \sqrt[6]{\frac{a}{b}\cdot\frac{a}{c}\cdot\frac{b}{a}\cdot\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{c}{b}} = 1$$

Thus, $\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b} \geq 6$, and so $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 3+6=9$, as requir

olution 2: By Cauchy-Schwarz, let $a_1=\sqrt{a}$, $a_2=\sqrt{b}$, $a_3=\sqrt{c}$, $b_1=\frac{1}{\sqrt{a}}$, $b_2=\frac{1}{\sqrt{b}}$, $b_3=\frac{1}{\sqrt{c}}$, and thus: $$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (1+1+1)^2 = 3^2 = 9,$$ and we are done.

Equality occurs iff $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$, i.e. iff $\underline{a=b=c}$.

] Find the maximum value of $x^3(4-x)$, where $0<x<4$.

Yuck. How can we use our knowledge of inequalities here? No really obvious way. That's why a little trickery is needed:

really nice trick here. See how the left side simplifies nicely?

Since $0<x<4$, we have by AM-GM, $$\frac{\frac{x}{3}+\frac{x}{3}+\frac{x}{3}+(4-x)}{4} \geq \sqrt[4]{\left(\frac{x}{3}\right)^3\cdot(4-x)}$$

$$\therefore \frac{x+(4-x)}{4} = 1 \geq \sqrt[4]{\frac{x^3(4-x)}{27}} \Rightarrow \underline{x^3(4-x) \leq 27.}$$ Max. value is 27.

Equality occurs iff $\frac{x}{3}=\frac{x}{3}=\frac{x}{3}=4-x$, i.e. if $\frac{4x}{3}=4$, or $\underline{x=3}$.

Checking, we see that if $x=3$, the maximum value of 27 is indeed attained.

4. If $a+b+c=1$, show that $(a+\frac{1}{a})^2+(b+\frac{1}{b})^2+(c+\frac{1}{c})^2 \geq \frac{100}{3}$, where $a, b, c > 0$.

By Cauchy, $\left[(a+\frac{1}{a})^2+(b+\frac{1}{b})^2+(c+\frac{1}{c})^2\right]\left[1^2+1^2+1^2\right] \geq \left[(a+\frac{1}{a})+(b+\frac{1}{b})+(c+\frac{1}{c})\right]^2$

$$= \left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

$$= \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2, \text{ since } a+b+c=1.$$

Using Cauchy again, we have $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c) \geq (1+1+1)^2 = 9$ ← see how we used Cauchy here?

Since $a+b+c=1$, that means $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$.

Hence, $\left[(a+\frac{1}{a})^2+(b+\frac{1}{b})^2+(c+\frac{1}{c})^2\right] \times 3 \geq (1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})^2 \geq (1+9)^2 = 100$, thus

$(a+\frac{1}{a})^2+(b+\frac{1}{b})^2+(c+\frac{1}{c})^2 \geq \frac{100}{3}$, as required.

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5. If the roots of the polynomial $x^6-6x^5+ax^4+bx^3+cx^2+dx+1$ are <u>all</u> positive, find a, b, c, and d.

There are two things you should be thinking: i) how is this an inequality problem?, and ii) surely there isn't enough information to figure this out! Check this out:

Let the roots of the polynomial be P_1, P_2, P_3, P_4, P_5, and P_6. We are given that $P_1, P_2, \ldots, P_6 > 0$. Also, $\begin{cases} P_1+P_2+P_3+P_4+P_5+P_6 = 6 \\ P_1P_2P_3P_4P_5P_6 = 1 \end{cases}$ (relationship between the roots of a polynomial and its coefficients)

By the AM-GM inequality, $\frac{P_1+P_2+P_3+P_4+P_5+P_6}{6} \geq \sqrt[6]{P_1P_2P_3P_4P_5P_6}$

works only because all the terms are non-negative

$$\therefore \quad 1 \geq 1.$$

Whoa, we have equality, i.e. $1=1$. That tells us that $P_1=P_2=P_3=P_4=P_5=P_6$.
from AM-GM

Since $P_1+P_2+\cdots+P_6=6$, that tells us that each term is equal to 1.

Hence all the roots of the polynomial are 1, so:

$$x^6-6x^5+ax^4+bx^3+cx^2+dx+1 = (x-1)^6$$
$$= x^6-6x^5+15x^4-20x^3+15x^2-6x+1$$

and matching up coefficients, we get: <u>$a=15$, $b=-20$, $c=15$ and $d=-6$</u>.

Isn't that a cool question?

6. Let A, B, and C be the angles of a triangle. Show that $\sin A+\sin B+\sin C \leq \frac{3\sqrt{3}}{2}$.

Here's where our buddy Jensen comes in handy. Let $f(x)=\sin x$. Then $f'(x)=\cos x$, and $f''(x)=-\sin x$. Since A, B, and C are angles of a triangle, $0^\circ < A, B, C < 180^\circ$. Also, $A+B+C=180^\circ$, but we'll use that later. For all x from 0° to 180°, $f''(x)<0$, and thus is concave in that interval.

Hence, since A, B, and C lie in this interval, by Jensen's Inequality,

$$\frac{f(A)+f(B)+f(C)}{3} \leq f\left(\frac{A+B+C}{3}\right)$$

$$\therefore \frac{\sin A+\sin B+\sin C}{3} \leq \sin\left(\frac{180^\circ}{3}\right)=\frac{\sqrt{3}}{2}$$, since $A+B+C=180^\circ$.

Multiplying both sides of the inequality by 3, we arrive at the desired result.

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7. Let a, b, and c be positive real numbers. Show that $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ (1995 CMO.

(There are several ways to do this, but this one is really ingenious).

Let $f(x)=(\ln x)\cdot x$. Then $f'(x)=\ln x+\frac{1}{x}\cdot(x)=\ln x+1$, and $f''(x)=\frac{1}{x}$, and $f''(x)>0$ for all positive values of x. Thus, by Jensen's Inequality, for positive a, b, and c, we have:

$$\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$$

$$\therefore \frac{a\ln a+b\ln b+c\ln c}{3} \geq \frac{a+b+c}{3}\cdot\ln\left(\frac{a+b+c}{3}\right)$$

$$\Leftrightarrow \frac{1}{3}\cdot\left(\ln a^a+\ln b^b+\ln c^c\right) \geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c}\cdot\frac{1}{3}$$

$$\Leftrightarrow \ln a^a b^b c^c \geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c}$$

$$\Leftrightarrow a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

By AM-GM, $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, thus $a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} \geq \left(\sqrt[3]{abc}\right)^{a+b+c}$

$$= (abc)^{\frac{a+b+c}{3}}.$$

Thus, we have proven the desired inequality.
(note: equality occurs iff $a=b=c$).

8. Suppose a, b, and c are the sides of a triangle. Show that:

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$$ (1996 APMO, last question).

Since a, b, and c are the sides of a triangle, $\begin{cases} a+b-c>0 \\ a+c-b>0 \\ b+c-a>0 \end{cases}$. Let $x=a+b-c$, $y=a+c-b$ and $z=b+c-a$. Then $x,y,z>0$ and we can express a, b, and c in terms of x, y, and z.

Our inequality then becomes: (i.e. is equivalent to) $\sqrt{x}+\sqrt{y}+\sqrt{z} \le \sqrt{\frac{x+y}{2}}+\sqrt{\frac{x+z}{2}}+\sqrt{\frac{y+z}{2}}$.

Since $x,y,z>0$, and thus, $\sqrt{x},\sqrt{y},\sqrt{z}>0$. So let's make another substitution: $x=p^2$, $y=q^2$, and $z=r^2$, where $p,q,r>0$. Then we need to prove that:

$$p+q+r \le \sqrt{\frac{p^2+q^2}{2}}+\sqrt{\frac{p^2+r^2}{2}}+\sqrt{\frac{q^2+r^2}{2}}.$$

(really neat use of symmetry here. Always try to exploit symmetr[y])

But by QM-AM, we have $\sqrt{\frac{p^2+q^2}{2}} \ge \frac{p+q}{2}$, $\sqrt{\frac{p^2+r^2}{2}} \ge \frac{p+r}{2}$ and $\sqrt{\frac{q^2+r^2}{2}} \ge \frac{q+r}{2}$.

Adding up these three inequalities, we get $\sqrt{\frac{p^2+q^2}{2}}+\sqrt{\frac{p^2+r^2}{2}}+\sqrt{\frac{q^2+r^2}{2}} \ge \frac{p+q}{2}+\frac{p+r}{2}+\frac{q+r}{2}=p+q+$[r] as desired. Thus, we are done.

] Suppose a, b, and c are all positive. Prove that $(a^3+b^3+abc)^{-1}+(b^3+c^3+abc)^{-1}+(c^3+a^3+abc)^{-1} \le (abc)^{-1}$
1998 USAMO, question #2

Here's a really nice trick to remember: if we replace a, b, and c by ka, kb, and kc, all the terms with k's will cancel out, and we'll get back to the original inequality → try it, you'll see that all the k's will disappear. Thus, we can assume without loss of generality that $\underline{abc=1}$!!! It makes things so much easier! Even if a, b, and c aren't numbers that multiply to 1, we can multiply all of them by a constant k so that the relation holds, so that's why we can do that.

Furthermore, we can let $x=a^3$, $y=b^3$, and $z=c^3$, since that will make the simplificati[on] easier. Since $abc=1$, we have $xyz=a^3b^3c^3=1$. So now our inequality becomes:

$$(x+y+1)^{-1}+(y+z+1)^{-1}+(z+x+1)^{-1} \le 1, \text{ where } x,y,z>0 \text{ and } xyz=1.$$

Much easier, isn't it? Well, after simplification, we get:

$$2(x+y+z) \le x^2y+xy^2+xz^2+x^2z+yz^2+y^2z = (x+y+z)(xy+yz+zx)-3xyz$$

$\therefore (x+y+z)(xy+yz+zx-2) \ge 3xyz=3$. That's what we need to prove.

And because $\frac{x+y+z}{3} \ge \sqrt[3]{xyz}=1$ and $\frac{xy+yz+zx}{3} \ge \sqrt[3]{x^2y^2z^2}=1$, by AM-GM, we have $(x+y+z)(xy+yz+zx-2) \ge 3\cdot(3-2)=3$, and so we are done.

10. Suppose that $a, b,$ and c are positive real numbers such that $abc=1$. Prove that: $\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$. (1995 IMO, Qu.#2)

The trick is to make the substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, and $c=\frac{1}{z}$. If you don't do this, the problem is virtually impossible to solve. See how that trick is useful?

Thus, we have $\frac{1}{\frac{1}{x^3}\left(\frac{1}{y}+\frac{1}{z}\right)} + \frac{1}{\frac{1}{y^3}\left(\frac{1}{x}+\frac{1}{z}\right)} + \frac{1}{\frac{1}{z^3}\left(\frac{1}{y}+\frac{1}{x}\right)} \geq \frac{3}{2}$

★ try to recognize substitutions like these that will give you an inequality that is easier to prove

$$\Leftrightarrow \frac{x^3yz}{y+z} + \frac{y^3xz}{x+z} + \frac{z^3xy}{x+y} \geq \frac{3}{2}$$

because $xyz=\frac{1}{abc}=1$.

$$\Leftrightarrow \frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2} \quad \text{(since } xyz=1\text{)}.$$

Hence, if we can prove this inequality, we will be done. There are now a couple of ways to proceed.

Method 1: (Cauchy).

Since $x,y,z \geq 0$, we have by Cauchy: $\left(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right)\left((y+z)+(x+z)+(x+y)\right) \geq (x+y+z)^2$

$$\Rightarrow \left(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right)\cdot(2x+2y+2z) \geq (x+y+z)^2$$

See how Cauchy is used here? Beautiful trick - exploits the symmetry of the expression

$$\Rightarrow \frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y} \geq \frac{x+y+z}{2} \geq \frac{3}{2}, \text{ by AM-GM.}$$

Thus we are done.

$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1$, so $x+y+z \geq 3$ by AM-GM.

Method 2: (Jensen)

Let $f(p) = \frac{p^2}{x+y+z-p}$. Then one can show that $f''(p) = \frac{2}{x+y+z-p} + \frac{2p(2x+2y+2z-p)}{(x+y+z-p)^3}$, and this is clearly positive if $0<p<x+y+z$. Thus, since $0<x,y,z<x+y+z$, by Jensen, we have

$$\frac{f(x)+f(y)+f(z)}{3} \geq f\left(\frac{x+y+z}{3}\right) \Rightarrow \frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y} \geq 3\cdot\frac{\left(\frac{x+y+z}{3}\right)^2}{(x+y+z)-\frac{x+y+z}{3}} = \frac{x+y+z}{2}.$$

By AM-GM, $x+y+z \geq 3$ (same as before), so $\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y} \geq \frac{3}{2}$, as required.

Remember, when you see an inequality problem, be clever and try to use some of the ideas detailed in these solutions. Who knows, that might be the way to do it! With problems like these, perseverance and tenacity is what you need - you might have to try 5 to 10 (or more!) different methods before you finally get it! ★

Some Fun Problems For You To Do

1. If a_1, a_2, a_3 are non-negative, show that $\frac{a_1}{2}+\frac{a_2}{3}+\frac{a_3}{6} \ge a_1^{\frac{1}{2}} a_2^{\frac{1}{3}} a_3^{\frac{1}{6}}$.

2. Show that if $a, b, c > 0$, then $(a+b)(a+c)(b+c) \ge 8abc$.

3. Find the greatest value of x^2y^3z given that $x, y, z > 0$ and $x+y+z=6$.

4. Show that if $a, b, c > 0$, then $(a+b+c)\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \ge \frac{9}{2}$. When does equality occur?

5. If $a, b > 0$, prove that $\frac{a+nb}{n+1} \ge \sqrt[n+1]{ab^n}$, where n is a positive integer. Using this, can you show that the sequence $(1+\frac{1}{n})^n$ is increasing?

Show that for all $n > 1$, we have $\left(\frac{n+1}{2}\right)^n > n!$. (note: we never have equality)

7. i) Prove that among all rectangles with a fixed perimeter P, the square has the greatest area.
ii) Prove that among all rectangles with a fixed area A, the square has the least perimeter.

Prove that among all triangles with a fixed perimeter P, the equilateral triangle has the greatest area.

Solve the system: $\begin{cases} 2^{x+y+z} = 64 \\ xyz = 8 \end{cases}$, where x, y, z are positive real numbers.

If A, B, and C are the angles of a triangle, show that $\cos A + \cos B + \cos C \le \frac{3}{2}$.

Suppose that $a_1, a_2, \ldots, a_n > 0$. Show that $\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots+\frac{a_n^2}{a_n+a_1} \ge \frac{a_1+a_2+\cdots+a_n}{2}$.

Suppose $a_1, a_2, \ldots, a_{1999}$ are 1999 positive real numbers, and let $b_1, b_2, \ldots, b_{1999}$ be some rearrangement of these numbers. What is the minimum value of $\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_{1999}}{b_{1999}}$?

Suppose that $a_1 \ge a_2 \ge \cdots \ge a_n > 0$, and that $b_1 \ge a_1$, $b_1b_2 \ge a_1a_2$, $b_1b_2b_3 \ge a_1a_2a_3, \ldots,$ $b_1b_2\cdots b_n \ge a_1a_2\cdots a_n$. Prove that $b_1+b_2+\cdots+b_n \ge a_1+a_2+\cdots+a_n$. (quite hard!)

Let n be an integer, $n \ge 3$. Let $a_1, a_2, \ldots, a_n$ be real numbers, where $2 \le a_i \le 3$ for $i=1,2,\ldots,n$. If $S = a_1+a_2+\cdots+a_n$, prove that:

$$\frac{a_1^2+a_2^2-a_3^2}{a_1+a_2-a_3}+\frac{a_2^2+a_3^2-a_4^2}{a_2+a_3-a_4}+\cdots+\frac{a_n^2+a_1^2-a_2^2}{a_n+a_1-a_2} \le 2S-2n.$$

(1995 IMO Shortlist Problem).

Prove the AM-GM inequality using Jensen's Inequality (hint: let $f(x) = \ln x$).

If you have any questions, please feel free to e-mail me!

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